Influence of hydrodynamic interactions between particles on the turbulent flow in a suspension

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Abstract

A model based on the theory of Felderhof and Ooms [4] is used to evaluate the effect of hydrodynamic interactions between particles in turbulent two-phase flows. The particles are assumed to be point-particles, obeying Stokes’ drag law. The flow of the suspending fluid phase is assumed to be (stochastic) stationary, homogeneous, isotropic turbulence. To simplify the calculations, only two-particle hydrodynamic interactions are taken into account. The value of the reduction function $J$, which is a measure of the hydrodynamic interactions, is evaluated as a function of the dimensionless wavenumber and frequency for a range of values of the density ratio between fluid and particles. Only for relatively heavy particles, hydrodynamic interactions affect the reduction function $J$ and the corresponding spectral transfer functions. However, it was found that the interactions can be neglected in turbulent flows, since very little energy is present at the frequencies where the interactions take place.

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1 Introduction

The difficulties in successfully modeling turbulent fluid-particles flows are caused by the complexity of the interaction between the two phases. At very

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low concentrations the influence of the particles on the surrounding fluid can be neglected. The single-phase fluid flow field is calculated and (representative) particles are tracked in this flow field using the particle equation of motion. However at somewhat higher concentrations one has to account for the two-way coupling effect: due to the inertia of the particles and their finite volume, the turbulent fluid flow field is changed. Direct numerical simulations with two-way coupling effect have been carried out by, for instance, Squires and Eaton [8] (stochastic stationary, isotropic turbulence) and by Elgobashi and Truesdell [3] (decaying, homogeneous turbulence). However these simulations are only possible at rather low concentration of the particles. At still higher concentrations the very large numbers of particles that are present in the flow field, make direct numerical simulations impossible.

To model flows at higher concentrations, a commonly used approach is to renounce the modeling of all individual particles and to model a mean effect on the fluid by the ensemble of particles. An example of this approach is the calculation by Batchelor [1] of the effective viscosity of a suspension to first order in the volume fraction of the particles. However his calculation is only valid for a stationary flow, not for a turbulent flow. Turbulence is, of course, not stationary and a range of frequencies occur in the flow field. In turbulence also inertial effects are important next to viscous effects. Therefore, an effective mass density has to be considered next to an effective viscosity. To that purpose Felderhof and Ooms [4] have studied the effective mass density of the suspension. They made a point particle approximation and took two-particle hydrodynamic interactions into account.

The question that might arise when considering the above, is how the effective suspension properties can be derived from the single-particle behavior. Even for a single particle, one already has to account for numerous effects like drag, buoyancy, virtual mass, Basset and Faxen forces. As a governing equation, the Basset-Boussinesq-Oseen (BBO) equation is often used. Already in 1949 Tchen [9] used this equation to study the motion of particles in turbulent flows. In their work both Batchelor [1] and Felderhof and Ooms [4] only took into account the viscous drag; so they used Stokes’ law.

The aim of this paper is to study (using the model of Felderhof and Ooms [4]) the influence of the hydrodynamic interactions (in two-particle approximation) between suspended particles on the energy density spectrum of a fluid and the particles in a suspension in homogeneous, isotropic turbulent motion. These interactions are defined as the forces that are exerted on a particle by another particle via the suspending fluid phase. One may realize the importance of the fluid inertia for the hydrodynamic interactions by considering the behavior of the Green’s function in the limit at large distances between two particles. (The Green’s function describes the hydrodynamic influence of one particle on the other.) Felderhof and Ooms [4] showed, that at any finite frequency the Green’s
function is dipolar \((1/r^3)\) at large distances, whereas at zero frequency it is identical to the Oseen tensor \((1/r)\). Mathematically, the dipolar character of the Green’s function changes the calculation (for turbulent flows) completely, as compared to flows without inertia effects.

In literature, the general analytical solution of the hydrodynamic interactions of \(n\) particles in Stokes flow is known, but the resulting equations are too complex to have practical use for turbulently flowing suspensions ([2],[5]). It is expected that a number of simplifications can be made.

2 Theory

A very brief description of the theory is given first. For a full derivation and application, one is referred to the original publication (Felderhof and Ooms [4] and subsequent papers (e.g. Ooms and Jansen [6]).

The fluid is assumed to obey the linearized Navier-Stokes equations with an external stochastic force acting on the fluid. The stochastic force density may be imagined to represent the effect of random sources, giving the fluid flow an energy spectrum distribution in frequency and wave number space. Particles are modeled by point forces acting on the fluid. The particles themselves obey Stokes’ drag law. Though their action on the fluid is modeled using the point-force approximation, the effect of their size is taken into account via the drag law and via the probability density function of the particle concentration distribution. The hydrodynamic interaction terms are grouped according to the number of particles involved and simplified using cluster expansion theory. The two-way coupling effect (including the hydrodynamic interactions) is expressed as an effective mass density.

The resulting equations, relating the energy density spectrum \(S_0\) of the fluid without particles to the energy density spectrum of the fluid with particles \(S_f\) and the particle spectrum \(S_p\) are given by:

\[
S_f \left(qa, \omega \tau, \phi, \rho_f/\rho_p, a/\Lambda, \eta_K/\Lambda\right) = |H(qa, \omega \tau, \phi, \rho_f/\rho_p)|^2 S_0(qa, \omega \tau, \rho_f/\rho_p, a/\Lambda, \eta_K/\Lambda) \tag{1}
\]

\[
S_p \left(qa, \omega \tau, \phi, \rho_f/\rho_p, a/\Lambda, \eta_K/\Lambda\right) = |\Gamma^T(qa, \omega \tau, \phi, \rho_f/\rho_p)|^2 \times |H(qa, \omega \tau, \phi, \rho_f/\rho_p)|^2 S_0(qa, \omega \tau, \rho_f/\rho_p, a/\Lambda, \eta_K/\Lambda) \tag{2}
\]

In these equations, \(q\) and \(\omega\) denote wavenumber and frequency, respectively.
\( a \) is the particle radius, \( \phi \) the particle volume fraction, \( \rho_f \) the fluid density and \( \rho_p \) the particle density. \( \tau = m/\zeta \) is the particle relaxation time, in which \( m = \frac{4}{3} \pi a^3 \rho_p \) is the particle mass and \( \zeta = 6 \pi \eta a \) the particle friction factor. \( \eta \) the fluid viscosity. \( \Lambda \) is the integral length scale of turbulence and \( \eta_K \) the Kolmogorov length scale. (In case \( S_0 \) is given as function of the turbulence energy per unit volume \( (k) \) and the dissipation per unit volume \( (\epsilon) \), \( \Lambda \) and \( \eta_K \) can be calculated with the well-known relations \( \Lambda = k^{3/2}/\epsilon \) and \( \eta_K = (\nu^3/\epsilon)^{1/4} \) in which \( \nu \) is the kinematic viscosity of the fluid.) The spectrum \( S_0 \) of the fluid without particles is, of course, not dependent on the particle radius \( a \) and the particle density \( \rho_p \). However by writing \( S_0 \) as function of the dimensionless wavenumber \( qa \) and the dimensionless frequency \( \omega \tau \) it becomes automatically also dependent on \( a/\Lambda \) and \( \rho_f/\rho_p \). The transfer functions \( H \) and \( \Gamma^T \) are specified by:

\[
\Gamma^T(qa, \omega \tau, \phi, \rho_f/\rho_p) = (1 - i \omega \tau(1 - \phi J(qa, \omega \tau, \rho_f/\rho_p)))^{-1}
\]

\[
H(qa, \omega \tau, \phi, \rho_f/\rho_p) = \frac{(\alpha a)^2 + (qa)^2}{(\alpha a)^2 + (qa)^2 - \frac{9}{2} \phi i \omega \tau \Gamma^T(qa, \omega \tau, \phi, \rho_f/\rho_p)}
\]

with \( (\alpha a)^2 = -\frac{9}{2}(\rho_f/\rho_p)i \omega \tau \). The hydrodynamic interactions, the main topic of this article, are described by the reduction function \( J(qa, \omega \tau, \rho_f/\rho_p) \); it is specified in the appendix. When \( J \) is omitted from the equations, the hydrodynamic interactions between the particles are left out of the calculations. The purpose of this publication is to calculate \( J \) as function of \( qa \) and \( \omega \tau \) for different values of \( \rho_f/\rho_p \) and to use the results to investigate the influence of the hydrodynamic interactions between the particles on the energy density spectrum of the fluid and on the energy density spectrum of the particles via equations (1) and (2). This is done by means of the following calculation procedure.

First a value for the density ratio \( \rho_f/\rho_p \) is chosen. Then the reduction factor \( J \) is calculated as function of the independent dimensionless variables \( qa \) and \( \omega \tau \). Next a value of the particle volume fraction \( \phi \) is chosen and the functions \( H \) and \( \Gamma^T \) are calculated as function of \( qa \) and \( \omega \tau \). Then the ratio of \( (|\Gamma^T|)^2 \) with hydrodynamic interaction and \( (|\Gamma^T|)^2 \) without hydrodynamic interaction \( (J = 0) \) is calculated as function of \( qa \) and \( \omega \tau \); and also the ratio of \( (|H|)^2 \) with hydrodynamic interaction and \( (|H|)^2 \) without hydrodynamic interaction as function of \( qa \) and \( \omega \tau \). It is obvious that the hydrodynamic interaction between the particles can only influence the turbulence of the suspension significantly at those values of \( qa \) and \( \omega \tau \), where these ratios are significantly different from 1. However for a significant influence on the turbulence it is also necessary, that at those values of \( qa \) and \( \omega \tau \) the energy density in the spectrum of the fluid without particles is significant. Therefore, the energy distribution for isotropic, homogeneous turbulence is studied by considering a representative spectrum and the corresponding relaxation function.
Fig. 1. the reduction factor $|J|$ for a density ratio $\rho_f/\rho_p$ of 100.

### 3 Calculation Procedure and Results

As mentioned, the reduction factor $J$ is dependent on the dimensionless groups $qa$, $\omega \tau$ and $\rho_f/\rho_p$. For a fixed density ratio $\rho_f/\rho_p$, the value of $J(qa, \omega \tau, \rho_f/\rho_p)$ was evaluated by numerical integration of the equations given in the appendix for the following ranges:

- $qa : 10^{-10} - 10^{10}$, 100 steps
- $\omega \tau : 10^{-8} - 10^{5}$, 110 steps
- $\rho_f/\rho_p : 10^{-3} - 10^{3}$, 30 steps

All the resulting data for $J(qa, \omega \tau, \rho_f/\rho_p)$ showed a similar shape (see Figure 1 and 2). For values of $\omega \tau$ and $qa$ smaller than 1, the value of $J$ is nearly constant and almost completely due to the $J_1$-integral in $J$ (i.e. the virtual overlap contribution). Only for low values of $\rho_f/\rho_p$, the $J_2$ integral (i.e. the hydrodynamic interactions) starts to influence the shape of $J$, in the region of $\omega \tau = 1$.

As illustration, the corresponding graphs for $|\Gamma^T|^2$ and $|H|^2$ are shown in Figure 3 (with $\rho_f/\rho_p = 0.01$). It can be seen that large eddies with low frequency are damped by the particles.

The influence of the hydrodynamic interactions of the particles is studied by plotting the ratios $|\Gamma^T|^2 / |\Gamma_{NI}^T|^2$ and $|H|^2 / |H_{NI}|^2$ for the density ratios 0.01, 1 and 100. The subscript $NI$ refers to the calculations were $J$ was set to

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\[ \text{the graphs only show approx. } 40 \times 40 \text{ data points for clarity.} \]
For the chosen volume load, $\phi = 0.05$, the influence of the hydrodynamic interactions on both transfer functions, $\Gamma^T$ and $H$, can be neglected in the case of $\rho_f/\rho_p = 100$, since the ratio of the results with and without interactions is close to unity. For the case with $\rho_f/\rho_p = 1$, there is a reasonable influence on the $\Gamma^T$ function, i.e. on the particle transfer function. For $qa < 1$, there is an increase of approximately 6% at $\omega\tau = 1$. In the case of heavy particles, $\rho_f/\rho_p = 0.01$, both transfer functions are affected significantly, in the same region. $\Gamma^T$ is increased up to 100%, while $H$ is decreased by approximately 50% when compared to the case without interactions (see fig. 6).

The question remains, whether this influence of the hydrodynamic interactions can be noticed. To answer this question, a general, representative turbulence
spectrum $E_0(q, \omega)$ is considered: the energy is spread over the wave numbers ranging from $q_E$ (the energy containing scale) to $q_K$ (the Kolmogorov scale) obeying the $'−5/3'$-law. Next to this, the energy is divided over the frequency domain following a relaxation function $R_0$:

$$E_0(q) = C q^{2/3} \nu^{5/3} f_L f_n$$

$$R_0(q, \omega) = \frac{1}{\pi} \frac{\nu q^2}{\omega^2 + (\nu q^2)^2}, \quad N.B.: \int_{-\infty}^{\infty} R_0(q, \omega) d\omega = 1;$$

$$S_0(q, \omega) = \frac{E_0}{4\pi q^2} R_0(q, \omega);$$

The functions $f_L$ and $f_n$ are structure functions, describing the shape of the spectrum. For a description, one is referred to [7]. For realistic values for the relevant parameters of the turbulence spectrum (e.g. macro length and velocity scale, Reynolds number, viscosity), it becomes clear that the hydrodynamic interactions have their influence in the range where energy is present: the influence was found for $qa < 1$ (or $q < 1/a$) and the energy is present between $q_E$ and $q_K$, which are generally both smaller than $q = 1/a$ for small particles.

When the relaxation function $R_0(q, \omega)$ is taken into account, it becomes obvious that in the region of interaction ($\omega \tau = 1$), the energy density is negligible when compared to the values close to $\omega = 0$: the function $R_0(q, \omega)$ decreases very rapidly for increasing $\omega$. In figure 7, a contour plot of an arbitrary, representative energy spectrum $E_0(q, \omega)$ is drawn. Also in this plot, the region of hydrodynamic interaction is shown (at $\omega \tau = 1$). From the contour lines, it can be seen that the energy density is very low near the interaction region. Therefore, the interactions can never have a significant effect on the fluid and particle spectrum, since the disturbed fluid and particle spectra are the product of transfer functions and reference (‘undisturbed’) spectrum.

### 4 conclusions

The influence of hydrodynamic interactions of point-particles have been studied using the model of Felderhof and Ooms. It was found that the influence on the transfer functions $\Gamma^T$ and $H$ increased for a decreasing density ratio. In the case of $\rho_f/\rho_p = 0.01$, both transfer functions were affected significantly. For $\rho_f/\rho_p = 1$, a minor effect on the particle transfer function was found, while effects on the fluid transfer function were negligible. For light particles (e.g. $\rho_f/\rho_p = 100$), no effects were found. When the influence of the interactions was linked with the energy that is present in isotropic, homogeneous turbulence, it...
was found that the interactions could not affect the fluid or particle spectra. The interactions occur at frequencies that are too high when compared to fluid turbulence.

Fig. 4. left: $|\Gamma_T|^2 / |\Gamma_{NI}^T|^2$ for $\rho_f/\rho_p = 100$; right: $|H|^2 / |H_{NI}|^2$ for $\rho_f/\rho_p = 100$

Fig. 5. left: $|\Gamma_T|^2 / |\Gamma_{NI}^T|^2$ for $\rho_f/\rho_p = 1$; right: $|H|^2 / |H_{NI}|^2$ for $\rho_f/\rho_p = 1$

5 Appendix A

Summary of the resulting equations, derived by Felderhof and Ooms [4]:

\[ J = J_1 + J_2 \]  
\[ J_1 = 3 \int_0^2 \left[ A_1(qax) \hat{f}_\alpha(x) + A_2(qax) \hat{g}_\alpha(x) \right] x^2 dx \]
Fig. 6. left: $|\Gamma|^2 / |\Gamma_{NI}|^2$ for $\rho_f/\rho_p = 0.01$; right: $|H|^2 / |H_{NI}|^2$ for $\rho_f/\rho_p = 0.01$

Fig. 7. a contour plot of $10\log E_0(\omega, \tau)$ and the influence on $\Gamma^T$, arbitrary units.

\[
J_2 = \lambda \int_0^\infty \frac{1 - 3\lambda A_1(qax) f_\alpha(x)}{1 - \lambda f_\alpha(x)^2} f_\alpha^2(x) dx
+ \frac{2 - 3\lambda A_2(qax) \tilde{g}_\alpha(x)}{1 - \lambda \tilde{g}_\alpha(x)^2} \tilde{g}_\alpha^2(x) x^2 dx
\]  
\[A_1(qax) = (qax)^{-3} [\sin(qax) - qax \cos(qax)] \]  
\[A_2(qax) = (qax)^{-1} \sin(qax) - A_1(qax)\]
\[ \hat{f}_n(x) = 3 \left( \frac{1}{(\alpha \alpha)^2 x^3} - \frac{1 + (\alpha \alpha) x}{(\alpha \alpha)^2 x^3} e^{-(\alpha \alpha) x} \right) \]  
(13)

\[ \hat{g}_n(x) = \frac{3}{2} \left( -\frac{1}{(\alpha \alpha)^2 x^3} + \frac{1 + (\alpha \alpha) x + (\alpha \alpha)^2 x^2}{(\alpha \alpha)^2 x^3} e^{-(\alpha \alpha) x} \right) \]  
(14)

\[ \lambda = -i \omega \tau / (1 - i \omega \tau) \]  
(15)

References


[9] Tchen, C.M. 1949 Mean values and correlation problems connected with the motion of small particles suspended in a turbulent fluid, *PhD thesis, Delft University of Technology*