DIRECT NUMERICAL SIMULATIONS OF PLANE CHANNEL FLOW OVER A 3D CARTESIAN GRID OF CUBES

W.P. Breugem*, B.J. Boersma* and R.E. Uittenbogaard**
* Laboratory of Aero and Hydrodynamics, J.M. Burgerscentrum
Leeghwaterstraat 21, 2628 CA Delft, The Netherlands
** WL| Delft Hydraulics, P.O. Box 177, 2600 MH Delft, The Netherlands
w.p.breugem@wbmt.tudelft.nl

ABSTRACT
Direct Numerical Simulations (DNS) have been performed of both laminar and turbulent flow in a plane channel with a solid top wall and a lower porous wall that consists of a 3D Cartesian grid of cubes. The simulations enable the direct evaluation of closures for the drag and the dispersion stress that are required to solve the Volume-Averaged Navier-Stokes (VANS) equations for flow in porous media. This paper presents a flavour of the DNS results for laminar channel flow. The drag closure proposed by Irmay [1] appears to be successful for flow through a grid of cubes with a porosity of 0.875. A variable-permeability model based on this closure does not perform well for the laminar channel flow over the grid of cubes. The analytical solution of Ochoa-Tapia and Whitaker [2] for the velocity profile, based on a momentum-transfer model for the interface region, gives a good agreement with the DNS results.

NOMENCLATURE

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INTRODUCTION

From a fluid-mechanic point of view a permeable wall layer is basically a layer inside which a flow takes place that can interact with the flow outside the layer. Examples are flow over plant canopies and flow in rivers with a bed of stones. The understanding of the influence of wall permeability on the flow dynamics and structure is crucial for the development of accurate models for for instance the exchange of oxide and carbon dioxide between a forest and the atmosphere or the erosion of a river bed.

A permeable wall layer can be modelled numerically in several ways. Hahn et al. [3] performed a DNS of the turbulent flow in a plane channel with a lower porous wall. The boundary conditions proposed by Beavers and Joseph [4] for laminar flow were used to allow for a slip velocity parallel to the wall, with a slip coefficient that has to be specified, whereas a no-penetration condition was imposed for the wall-normal velocity. Another approach is followed by Breugem and Boersma [5] who performed a DNS of the same flow, but they used a continuum...
approach for the flow in the porous wall. In a volume-averaged sense the flow in the wall is governed by the Volume-Averaged Navier-Stokes (VANS) equations [6]. A major advantage of this approach is that no boundary conditions need to be specified at the interface with the porous wall. The main difficulty that is encountered in this approach is that in order to solve the VANS equations closures are required for the dispersion stress and for the drag force that the solid phase exerts on the fluid phase. This problem motivated the approach taken in the current study in which the flow in a plane channel is considered with a solid top wall and a lower porous wall that consists of a 3D Cartesian grid of cubes. In the DNS the complete flow field is resolved and this enables a direct evaluation of the closures that are required for the continuum approach.

The paper is organized as follows. First a brief introduction is given to the volume-averaging method and the VANS equations for the volume-averaged flow are given. The next section deals with the closure problem for the drag that the flow experiences in a 3D Cartesian grid of cubes. Thereafter a description is given of the numerical method that is used. Separate simulations of uniform flow through a grid of cubes have been performed to evaluate the closure for the drag. Results are then shown for laminar channel flow over the same grid of cubes. The last section gives a short summary and a discussion.

VOLUME-AVERAGING METHOD

The volume-averaging method provides the theoretical basis for a continuum approach of the flow in a porous medium. In this method the flow is averaged over a small spatial volume. The filtered flow field is defined in the fluid as well as in the solid phase. The volume-averaged flow is defined mathematically as:

$$\langle u \rangle_x = \frac{1}{V} \int_V y(r) m(r-x) u(r) \, dV$$

where the brackets <> denote the superficial volume average, V is the averaging volume, y is the phase-indicator function that equals 1 when the vector r points into the fluid phase and equals 0 when r points into the solid phase, and m is the weighting function.

The weighting function must satisfy the normalization condition:

$$\int_V m(r-x) \, dV = 1$$

In principle the weighting function can be chosen freely, but it is desirable that the filtered flow field has negligible variations inside the averaging volume. For an ordered porous medium, like a Cartesian grid of cubes that is considered in the present study, Quintard & Whitaker [7] proposed the following weighting function:

$$m(x) = \frac{1}{l_1^2 l_2 l_3} G(x_1, l_1) G(x_2, l_2) G(x_3, l_3)$$

in which l is the length of the unit cell in direction i, and G is defined according:

$$G(x, l) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Fig. 1 shows the averaging volume and the weighting function, which is referred to as the cellular-average filter.

![Figure 1: The averaging volume and filter for the cellular average of flow in an ordered porous medium.](image)

The velocity at a certain point in the porous medium can be decomposed into a contribution from the volume-averaged velocity at this point and a spatial deviation thereof:

$$u = \langle u \rangle / \epsilon + \hat{u}$$

in which $\epsilon$ is the porosity. According to equation (1) the porosity is defined as:

$$\epsilon = \int_V \gamma(r) m(r-x) \, dV = \frac{V_f}{V}$$

where $V_f$ is the volume of the fluid phase inside the averaging volume V.

Application of the spatial filter (1) to the Navier-Stokes equations yields the VANS equations [6]:

$$\begin{align*}
\frac{\partial \langle u \rangle}{\partial t} + \nabla \left[ \frac{\langle uu\rangle}{\epsilon} + \nabla \langle uu\rangle - \frac{\langle u\rangle \langle u\rangle}{\epsilon} \right] = & -\frac{1}{\rho} \nabla \cdot \langle p \rangle + \nu \nabla^2 \langle u \rangle + \\
\int_A m \n \left[ I P^2 + \nu \nabla \langle u \rangle \right] dA
\end{align*}$$

$$\nabla \cdot \langle u \rangle = 0$$

In equation (6) A is the contact area between the fluid and the solid phase inside the averaging volume V, and $\n$ is the normal vector at A that points from the fluid into the solid phase. Irrespective the filter that is used, the VANS equations as given above are exact for Newtonian and
incompressible flow in a porous medium with a time-independent porosity. The third term at the left-hand-side
represents transport of momentum by dispersion, and the last term at the right-hand-side corresponds to the drag
force per unit mass that the solid phase exerts on the fluid phase. In the special case that the porosity equals 1, the
drag integral vanishes, and the VANS equations are the same as the equations used in a Large Eddy Simulation
(LES) for a turbulent flow [8]. The dispersion stress in a LES originates in the presence of subgrid-scale eddies
that affect the trajectories of fluid elements. As a consequence, in the presence of mean shear, the subgrid
eddies cause a net transport of momentum. In porous media an additional contribution to dispersion originates in
the topology of the solid phase. For a low characteristic Reynolds number or a small ratio of d/ds (length scale
pores over diameter solid particles, see Fig. 1) topological dispersion will dominate strongly over turbulent
dispersion.

The VANS equations (6) can be simplified when the filter length l is much smaller than the typical length scales
L_u and L_v which respectively the volume-averaged flow and the porosity vary, i.e.:

\[ \frac{1}{L_u} \ll 1 ; \quad \frac{1}{L_v} \ll 1 \]  \hfill (8)

When these conditions hold, the dispersion term and the drag integral can be approximated by:

\[ \nabla \left[ \left( u \cdot u \right) - \frac{\left\langle u \cdot u \right\rangle}{\epsilon} \right] \approx \nabla \left\langle \hat{u} \cdot \hat{u} \right\rangle \]  \hfill (9)

\[ \int_A \frac{m}{\epsilon} \left[ -\frac{P}{\rho} + \nu \nabla u \right] \, dA \approx \frac{\nabla \epsilon}{\rho} \left\langle p \right\rangle \]  \hfill (10)

In general the approximations for the dispersion term and the drag integral are not valid at the interface between a
porous medium and a clear fluid domain. In this region both the porosity and the volume-averaged flow field
change rapidly in space, and consequently the length scale constraints (8) are violated. This complicates the
development of closures for the dispersion stress and the drag integral in terms of the volume-averaged velocity,
that are needed to solve the VANS equations.

**CLOSURE PROBLEM**

In this section a discussion is given of the closure problem for the drag that the flow encounters in a 3D
Cartesian grid of cubes, see Fig. 2. A discussion of the closure problem for the dispersion stress is omitted in this
paper.

For a stationary and uniform volume-averaged flow, the VANS equations (6) reduce to:

\[ 0 = -\frac{1}{\rho} \nabla \left\langle p \right\rangle + \int_A \frac{m}{\epsilon} \left[ -\frac{P}{\rho} + \nu \nabla u \right] \, dA \]  \hfill (11)

Whitaker [6] showed that the drag integral can be replaced by:

\[ \int_A \frac{m}{\epsilon} \left[ -\frac{P}{\rho} + \nu \nabla u \right] \, dA = -\nu K^{-1} \left( I + F \right) \left\langle u \right\rangle \]  \hfill (12)

where \( K, I \) and \( F \) are respectively the permeability, unit and Forchheimer tensor. In general the Forchheimer
tensor depends on the Reynolds number, on the geometrical parameters of the porous medium and also on the
orientation of the porous medium w.r.t. the direction of the volume-averaged flow. For very small Reynolds
numbers the Forchheimer tensor can be neglected and in this limit equation (11) reduces to Darcy’s Law. The
permeability tensor is a symmetric tensor of second order rank \[9\]. The 3D Cartesian grid of cubes falls in the class of
spherically isotropic porous media \[10\] for which the permeability tensor is isotropic: \( K = K \mathbf{I} \). To derive
expressions for the permeability and the Forchheimer tensor we consider the cases of a very low and a very
high porosity.

![Figure 2: 3D Cartesian grid of cubes. The volume-averaged flow is parallel to the x-axis.](image)

In the limit of a porosity close to zero, the ratio \( d/d_s \) is very small. The flow between positions \( x_0 \) and \( x_1 \),
see Fig. 2, can be approximated by plane channel flow, and the pressure drop between positions \( x_0 \) and \( x_2 \) can
be neglected. When in addition the characteristic Reynolds number is very low the flow is laminar. Based on these
assumptions Lam [1] derived an expression for the permeability of fissured rocks:

\[ K = \frac{1 - \left( 1 - \epsilon \right)^{1/3}}{12(1 - \epsilon)} \frac{d_s^2}{\epsilon^2} \]  \hfill (13)

in which the porosity is defined as:

\[ \epsilon = 1 - \frac{1}{\left( 1 + d_t/d_s \right)^3} \]  \hfill (14)
For $\varepsilon \ll 1$ the Forchheimer tensor is approximately isotropic: \( \mathbf{F} = \mathbf{F}_i \). An expression for \( F \) is found by the assumption of fully turbulent channel flow. The pressure drop in a hydraulically smooth channel with height \( d \), and bulk velocity \( U_b \) is approximately equal to \([11]\):

\[
\frac{\partial p}{\partial x} = -0.073 \frac{\rho U_b^2}{d} \left( \frac{U_b d}{\nu} \right)^{-1/4}
\]

(15)

This leads to the following estimate for \( F \):

\[
F = \frac{0.073}{12} \left( \frac{\langle u \rangle d_x}{\nu} \right)^{3/4}, \quad \varepsilon \ll 1
\]

(16)

In the limit of a porosity close to one, the flow near a cube is not influenced by the presence of other cubes and a closure of the drag can be estimated from the drag force \( D \) for flow around a single cube:

\[
D = -d_s^2 \frac{1}{2} \rho U^2 f
\]

(17)

in which \( f \) is the friction factor, which is a function of the Reynolds number \( Re = Ud / \nu \) and the orientation of the cube w.r.t. the direction of the far-field velocity \( U \). The friction factor must be determined from experiments or numerical calculations. In the limit of very small Reynolds numbers the friction factor is equal to \([12]\):

\[
f = C / Re, \quad Re \ll 1
\]

(18)

in which \( C \) is a constant. This yields for the permeability:

\[
K = C d_s^2 \left( \frac{\varepsilon}{1 - \varepsilon} \right), \quad 1 - \varepsilon \ll 1
\]

(19)

The expression for the Forchheimer component \( F_{xx} \) becomes:

\[
F_{xx} = C \left[ \frac{\langle u \rangle d_s}{\nu} \right] f, \quad 1 - \varepsilon \ll 1, \quad Re \gg 1
\]

(20)

For \( 1 - \varepsilon \ll 1 \) the Forchheimer tensor is definitely not isotropic, as the friction factor depends on the orientation of the cube w.r.t. the flow in the far field. For flow around a single sphere the friction factor is almost constant over a large range of Reynolds numbers \([13]\) and this behavior is also expected for a cube, which then would imply that \( F_{xx} \) is proportional to \( \langle u \rangle d / \nu \). This is different from the Reynolds number dependence for the case of very small porosity, see equation (16).

For the case that the porosity is neither very small nor very large, the pressure loss between \( x_i \) and \( x_s \) (see Fig. 2) can not be neglected. It is difficult to quantify this loss in a simple manner. Hence detailed experiments or numerical calculations are required to determine the permeability and the Forchheimer tensor for the range of porosities \( 0 < \varepsilon < 1 \).

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**NUMERICAL METHOD**

In this section the numerical method is discussed that is used in the DNS of plane channel flow over a 3D Cartesian grid over cubes. A sketch of the geometry is given in Fig. 3.

The full Navier-Stokes equations are solved on a fully staggered and uniform Cartesian computational grid. For the spatial discretization of the equations the finite-volume method is used with the second-order central-differencing scheme. The following second-order pressure-correction scheme is used:

\[
\frac{\bar{u}_i - u_i^n}{\Delta t} = \frac{5}{4} f_i^n - \frac{1}{4} f_i^{n-2} - \left[ \frac{\partial p}{\partial x_i} \right]_c
\]

(21a)

\[
\frac{\partial \bar{u}_i}{\partial x_i} = \frac{\partial^2 \bar{p}}{\partial x_i^2}
\]

(21b)

\[
u_i^{n+1} = \bar{u}_i - \Delta t \frac{\partial \bar{p}}{\partial x_i}
\]

(21c)

\[
p^{n+1} = p^n + \bar{p}
\]

(21d)

where the bar \(^\wedge\) refers to the prediction velocity and the hat \(^\wedge\) to the correction pressure. All quantities are made dimensionless with help of the bulk velocity in the channel \( U_b \) and the channel height \( H \). The function \( f \) in (21a) is given by:

\[
f_i = - \frac{\partial p}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j^2}
\]

(22)

where \( Re = U_b H / \nu \). The additional pressure gradient at the right-hand-side of (21a) is constant and used to drive the flow. In the DNS a finite flow domain is used and periodic boundary conditions are applied to the horizontal flow directions.

For the implementation of the cubes the Immersed Boundary Method (IBM) \([14]\) is used. In this method forces
are added to the flow field (i.e. the right-hand-side of equation (22)) to accomplish a zero velocity at the surface of the cubes. This is illustrated in Fig. 4.

![Figure 4](image)

**Figure 4:** Sketch of the computational grid and the forces \(D\) that in the IBM are added to the flow field.

The main benefits of the IBM are a relatively easy implementation of boundary conditions and the ability to use fast numerical algorithms as the flow field is also defined inside the cubes. The particular IBM that is used here, is similar to the one developed by Fadlun et al. [14]. Different from Fadlun et al. the cubes in our DNS are aligned along the computational grid such that the cube surface lies on the grid points for the normal (w.r.t. the cube surface) velocity. This enables an exact implementation of the no-slip boundary condition at the cube surface. This means for instance that the force \(D\) in Fig. 4 can be calculated exactly in order that the flow at this point experiences a no-slip velocity at the cube surface:

\[
\mathbf{u} \cdot \lambda = 0 \quad (23a)
\]

where \(\lambda\) is tangential unit vector. In the IBM the no-penetration condition for the normal velocity at the cube surface is approximately satisfied:

\[
\mathbf{u} \cdot \mathbf{n} = -dt \nabla \hat{p} \cdot \mathbf{n} \approx 0 \quad (23b)
\]

By updating the pressure every time step (see equation (21d)) the correction pressure remains very small and consequently the right-hand-side of (23b) is close to zero. For the special case of stationary flow the correction pressure is even zero.

The use of the IBM has consequences for the stability of the numerical scheme. The forces that are added to the flow field represent the drag force that the flow encounters by the presence of cubes. From equation (12) it can be deduced that this can cause large negative shifts of the real part of the eigenvalues of \(\hat{\xi}\), although it is difficult to quantify this in a simple manner. This is the reason for the choice of our time integration scheme (see equation (21a)) with coefficients \(5/4\) at time step \(n\) and \(-1/4\) at time step \(n-2\), as the stability curve of this scheme crosses the real axis at a much more negative value as compared to for instance the second-order Adams-Bashfort scheme [15]. The time step constraints used in the DNS are based on the constraints derived by Wesseling [15] for the second-order Adams-Bashfort scheme, which are multiplied by a factor 0.75.

**EVALUATION OF THE CLOSURE FOR THE DRAG**

In the DNS of channel flow over a 3D Cartesian grid of cubes, the ratio \(d/d_f\) (see Fig. 3) is taken equal to 1. This corresponds to a high value of the permeability for which a significant influence can be expected on the flow in the channel. Furthermore for \(d/d_f=1\) and a uniform computational grid, the amount of grid points along a cube is equal to the number of grid points in between two cubes. Intuitively this seems to be a good choice as both the boundary layers along the side walls of the cubes as well as the wakes in between the cubes must be resolved. To determine the permeability of the grid of cubes separate simulations are performed of a uniform flow (in a volume-averaged sense) through the grid of cubes and this is compared to the models given in an earlier section. The numerical method is the same as the one described in the last section, except that now also a periodic flow condition is used for the \(z\)-direction. The number of cubes is \(2 \times 2 \times 2\). Multiple simulations have been performed in which the Reynolds number (\(Re=\rho u d / \nu\)) is varied. Dependent on the Reynolds number the number of grid points is varied between \(80^3\) and \(104^3\). Fig. 5 shows a cross-section of the calculated flow field at \(Re=205\).

![Figure 5](image)

**Figure 5:** Cross-section of the flow field at \(Re=205\) (1 out of 4 vectors is shown).

The bulk velocity at this \(Re\) number is slightly instationary, but from Fig. 5 it can be seen that the flow is essentially laminar. Even at this high porosity of 0.875 the assumptions of Irma [1] for the derivation of equation (13) for the permeability appears to be fairly correct. In between \(z/d_f=1.5\) and \(z/d_f=2.5\) the velocity profile is parabolic, and in between \(x/d_f=1.5\) and \(x/d_f=2.5\) the velocity gradients in the vertical direction are small.
Fig. 6 shows the dimensionless pressure drop as function of the Re number. Each dot corresponds to a separate simulation.

![Figure 6: Dimensionless pressure drop as function of the Reynolds number.](image)

At Re=205 the dimensionless pressure drop has a local minimum. This minimum corresponds to the transition from laminar to turbulent flow at higher Re numbers. Figure 7 shows a cross-section of the flow field at Re=380.

![Figure 7: Cross-section of the flow field at Re=380 (1 out of 4 vectors is shown).](image)

Between x/d_s=1.5 and x/d_s=2.5 large vortical structures are present in wake regions of the cubes, which cause an exchange of momentum with the flow between z/d_s=1.5 and z/d_s=2.5. As a consequence, different from the case of laminar flow in Fig. 5, the pressure loss between x/d_s=1.5 and x/d_s=2.5 cannot be neglected.

The relation between the pressure drop and the velocity <u> is given by equations (11) and (12). The permeability K is obtained from the slope of the straight line in Fig. 6:

\[ K = 0.134d_s^2 \]  

(24)

This value is close to the prediction by equation (13) of 0.125d_s. More simulations in the range of high Re numbers (> 400) need to be performed to determine the value of the Forchheimer component F_{xx}. A complication is that the higher the Re number the smaller the smallest eddies, and hence a strongly increasing number of computational grid cells is required. With a DNS the range of Re number that can be calculated is therefore quite limited.

**Laminar Channel Flow Over a Grid of Cubes**

A DNS of laminar channel flow over a grid of cubes is performed. The flow configuration is shown in Fig. 3. The Reynolds number based on the bulk velocity in the channel and the channel height is fixed at Re=U/ch/v=1. We are aware that for Re=1 the use of the Navier-Stokes equations in the DNS is redundant as the flow is governed by the Stokes equations, but for Re=1 a relatively coarse computational grid was sufficient to obtain an accurate solution within a short time, and this was helpful to test the DNS code. Another reason for the choice of Re=1 originated in the aim to evaluate some available momentum-transfer models for Stokes flow over a porous medium. This will be discussed later on in this section.

![Figure 8: Porosity profile. The black cubes mark the location of the cubes.](image)

The flow in between the cubes is periodic by a displacement over the length of a unit cell, therefore it suffices to have only one cube in the horizontal directions and to use periodic boundary conditions. In the z-direction 7 cubes are used to ensure that in the core region of the porous medium the volume-averaged flow is uniform. The ratio d_s/H equals 0.0625. The number of grid points equals 28x28x448.

To obtain the volume-averaged flow field, at every grid point a second-order accurate discretized version of equation (1) is evaluated with the cellular-average filter as given by equation (3) and shown in figure 1. The same filter is used both for the porous medium and for the channel. The profile of the porosity, which is defined by equation (5), is shown in Fig. 8. The porosity is used to define the channel height H, which is equal to the distance between the top wall and the point at which the porosity...
Ochoa-Tapia and Whitaker obtained an analytical solution where starts to deviate from 1 (see averaging volume 1 in Fig. 3). In the interface region between the channel and the porous medium the porosity is varying of over length of $L_e = 3d$.

Fig. 9 presents the profile of the volume-averaged velocity (solid line) together with approximate solutions based on the model of Ochoa-Tapia and Whitaker [2] (OTW model). In the OTW model the permeability in the porous medium is taken constant. This is not correct in the interface region ($z = [-3d_0, 0]$) as in this region the permeability rises continuously till it approaches infinity at $z = 0$. To compensate this error, the OTW model introduces a jump condition in the viscous shear stress at $z = z_0$ where $z_0$ is located somewhere in the interface region:

$$\left[ \frac{\mu}{\epsilon} \frac{d}{dz} \langle u \rangle \right] \bigg|_{z = z_0} = \beta \left[ \frac{\mu}{\sqrt{K}} \langle u \rangle \right] \bigg|_{z = z_0},$$

where $\beta$ is a dimensionless coefficient, that has to be determined from experiments. The choice of $z_0$ is a bit arbitrary, but for the type of filter that is used here Ochoa-Tapia & Whitaker (p. 2652) suggested to locate $z_0$ at the centre of the interface region ($z = -3d/2$). When the permeability and porosity are constant and the flow is unidirectional the VANS equations (6) reduce to:

$$0 = -\frac{\partial \langle p \rangle}{\partial x} + \frac{\epsilon}{\epsilon} \frac{\partial^2 \langle u \rangle}{\partial z^2} - \frac{\omega}{K} \langle u \rangle,$$

Ochoa-Tapia and Whitaker obtained an analytical solution for the velocity profile by using equation (26) for the flow in the porous medium together with the Stokes equations for the flow in the channel. The solutions for the velocity in both regions were matched by equation (25) and the condition that the volume-averaged velocity $\langle u \rangle$ is continuous over the interface. The analytical solution predicts a parabolic profile in the channel and an exponentially decreasing function in the porous medium.

Fig. 9 shows that the OTW solution overlaps almost entirely over the DNS solution, except in the interface region where the velocity is underestimated. This is inherent to the assumption in the OTW model that the permeability is constant in this region. The choice for $z_0 = -3d/2$ gives a significantly better agreement than $z_0 = 0$. The value of $\beta$ in equation (25) is determined from the ratio of the bulk velocity $U_b$ over the constant creep velocity in the core region of the porous medium: $\beta = 0.45$ for $z_0 = -3d/2$ and $\beta = 0.91$ for $z_0 = 0$. These values are in agreement with the estimate made by Ochoa-Tapia and Whitaker that $\beta$ is of order 1, but they also show that $\beta$ is sensitive to the choice of the interface position $z_0$.

In Fig. 10 the budgets in the momentum equation (26) are shown. The last term in (26) contains a contribution from the pressure drag and from the viscous drag, see equation (12), and they are shown separately. In the channel ($z > 0$) the pressure gradient is in balance with the diffusion term. In the interface region, just below $z = 0$, the diffusion term becomes positive. This represents the diffusion of momentum into the porous medium. The diffusion of momentum plus the applied pressure gradient is in balance with the drag. Viscous drag dominates over pressure drag throughout the porous medium. In the core region of the porous medium, roughly for $z/H < 0.25$, the flow is uniform and the diffusion term is zero.

From the calculated drag the permeability is determined by assuming the validity of equation (12) in which the Forchheimer tensor can be neglected as the Reynolds number is very small. The profile for the inverse of the permeability is shown as the solid line in Fig. 11. The dashed line is a prediction based on equation (13) in which the local value of the porosity (see Fig. 8) is used and in which the coefficient 12 is replaced by 11.2 to agree with equation (24) when $\epsilon = 0.875$. The permeability appears to vary in the interface region over roughly 4
cube diameters. This is larger than the region over which the porosity is varying, which is equal to \(3d\_v\) (see averaging volume 3 in Fig. 3). Apparently the permeability does not only depend on the porosity, and this explains partly why the variable-permeability model does not perform very well. In fact, the permeability varies over the same thickness as the volume-averaged velocity, and this suggests a dependency of the permeability on the velocity gradient. Another reason why the variable-permeability model fails is that the topology at the interface is different from the topology of the core region when it would have the same porosity. This is clear from the comparison between averaging volumes 2 and 4 in Fig. 3.

![Profile of the dimensionless inverse of the permeability \(d^2_v/K\). Solid line: DNS. Dashed line: equation (13).](image)

**SUMMARY AND DISCUSSION**

DNS results have been presented of channel flow over a 3D Cartesian grid of cubes. The grid of cubes mimics a porous wall. The closure problem is discussed for the drag integral in the VANS equations. The DNS data is used to evaluate the closure for the drag and two momentum-transfer models for laminar flow over a porous wall. The OTW model gives good agreement with the DNS, but a disadvantage is that a value for the stress-jump parameter must be specified which is unknown a priori. A variable-permeability model in which the permeability is related to the local porosity in the interface region does not perform well. The results show that the permeability depends not only on the porosity but also on the gradient in the volume-averaged velocity. It is a challenge to develop new models for the variation of the permeability in the interface region.

Many porous media, like for instance a forest, have a complex geometry. This enforces the use of continuum models based on the volume-averaging method. DNS of a strongly simplified but representative porous medium is a good instrument to gain insight in the flow through the porous medium and the interaction with the outer flow field. It enables the validation and development of closures for the drag and the dispersion stress that are needed in the continuum models.

**REFERENCES**